Counterfactuals and Mediation

Brady Neal

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Counterfactuals Basics

Important Application: Mediation

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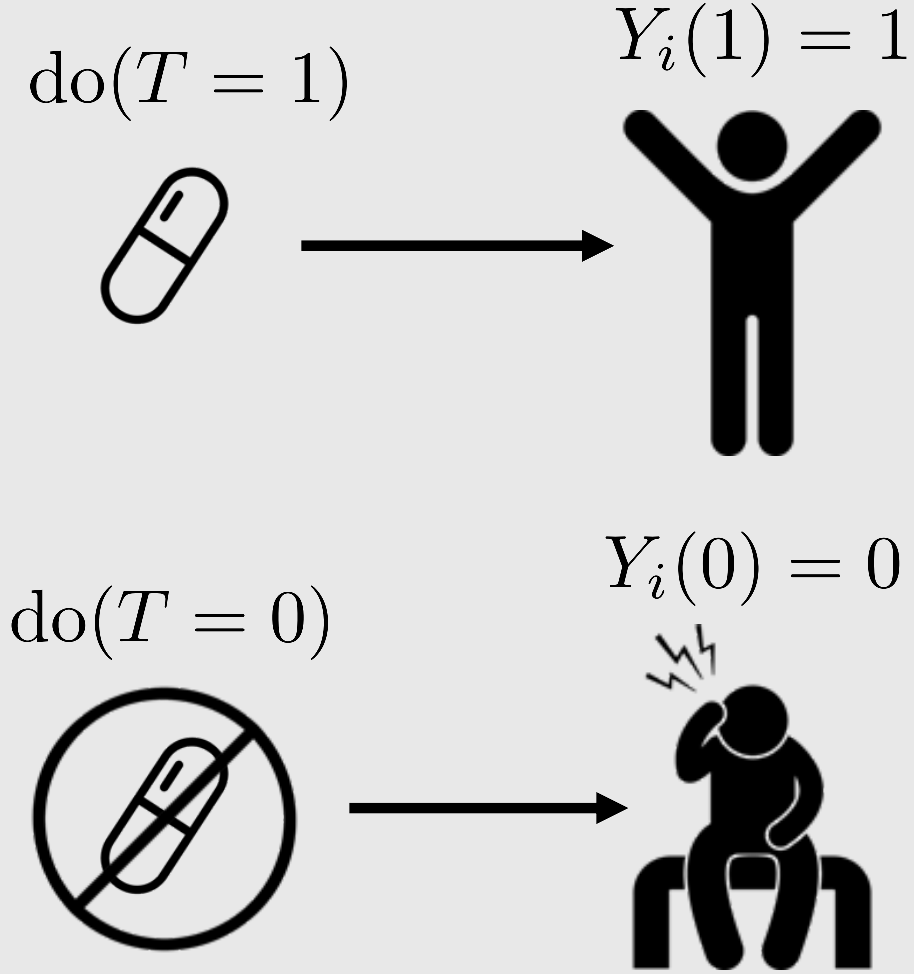
Important Application: Mediation

Fundamental Problem of Causal Inference



T	: observed treatment
Y	: observed outcome
i	: used in subscript to denote a specific unit/individual
$Y_i(1)$	: potential outcome under treatment
$Y_i(0)$	: potential outcome under no treatment

Causal effect

$$Y_i(1) - Y_i(0) = 1$$

Fundamental Problem of Causal Inference

Counterfactual

$\mathrm{do}(T=1)$ $\quad$ $Y_i(1) = 1$

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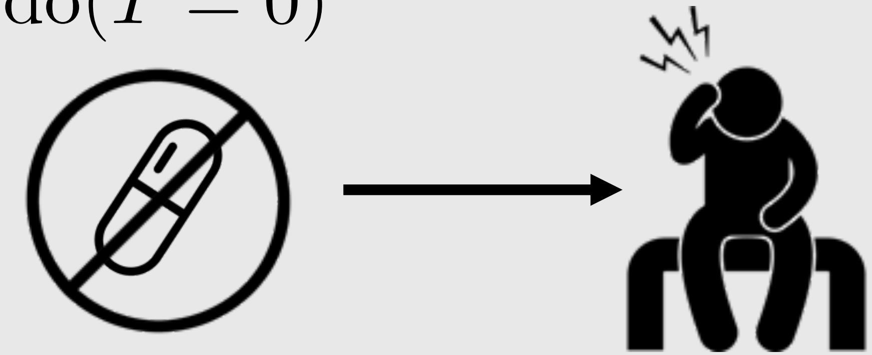
Factual

$\mathrm{do}(T=0)$ $\quad$ $Y_i(0) = 0$



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Fundamental Problem of Causal Inference

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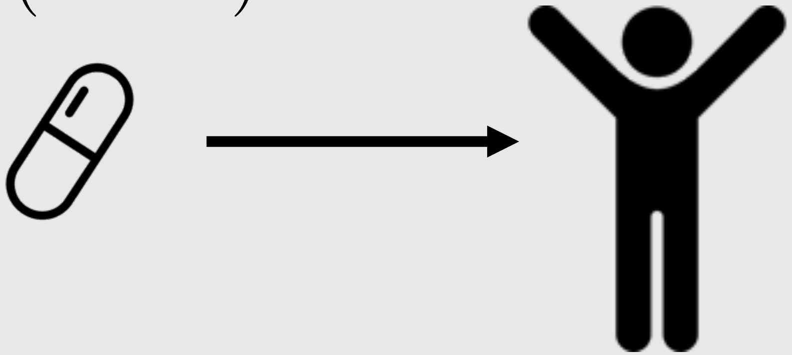

Counterfactual

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Causal effect

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We can compute counterfactuals using a parametric SCM.

Counterfactuals

Counterfactual: $P(Y(t) \mid T = t', Y = y')$

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Cannot express counterfactuals using do-notation

Roadmap for Computing Counterfactuals

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Given: Observation of (T, Y) (observation of potential outcome $Y(t)$ where t is the observed value of T)

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Main ingredient necessary: correct parametric model for the structural equation for Y

Result: access to counterfactuals Y(t') at the unit-level

Computing Counterfactuals: Simple Example

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SCM: $$\begin{aligned} T &:= \dots \\ Y &:= UT + (1 - U)(1 - T) \end{aligned}$$

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Step 1: Solve for U

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$Y = UT + (1-U)(1-T)$ $\quad Y_u(1)?$

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Step 2: Individualized SCM

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Step 2: Individualized SCM

$$T := 1$$

$$Y := T$$

$$Y_u(1) = 1$$

ITE: $Y_u(1) - Y_u(0) = 1 - 0 = 1$

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3. Prediction: Use the value of U from step 1 and the modified SCM from step 2 to compute the value of Y(t)

Question:

Given the observation T = 1 and Y = 0, compute Y(0) for this individual given the following SCM:

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Observation: T = 1 and Y = 0

$$(Y_u(1) = 0)$$

$$P(U = \text{never happy} \mid T = 1, Y = 0) = \frac{0.2}{0.2 + 0.1} = \frac{2}{3}$$

$$P(U = \text{dog-hater} \mid T = 1, Y = 0) = \frac{0.1}{0.2 + 0.1} = \frac{1}{3}$$

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$$Y_u(0) = ?$$

$$P(Y_u(0) = 1) = \frac{1}{3}$$

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No Unit-Level Counterfactuals without Parametric Model

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Strong assumption

No Unit-Level Counterfactuals without Parametric Model

Main ingredient necessary for computing counterfactuals: parametric model for the structural equation for Y

Strong assumption

Without it, we are stuck with the fundamental problem of causal inference.

Question:

Given the observation T = 1 and Y = 1, compute Y(0) for this individual given the following SCM and prior:

$$Y := \begin{cases} 1 & U = \text{always happy} \\ 0 & U = \text{never happy} \\ T & U = \text{dog-needer} \\ 1 - T & U = \text{dog-hater} \end{cases}$$

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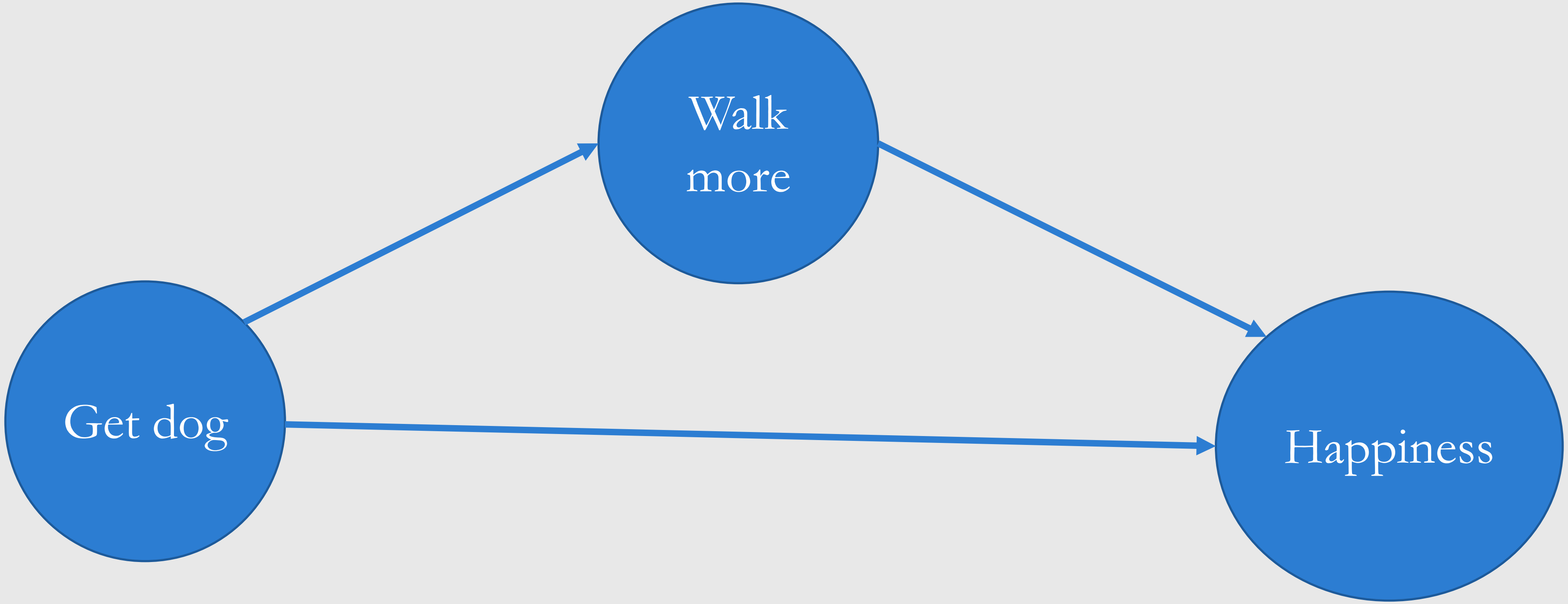
See Malinsky et al. (2019)'s potential outcome calculus (generalization of do-calculus) for general identification of counterfactual quantities

Counterfactuals Basics

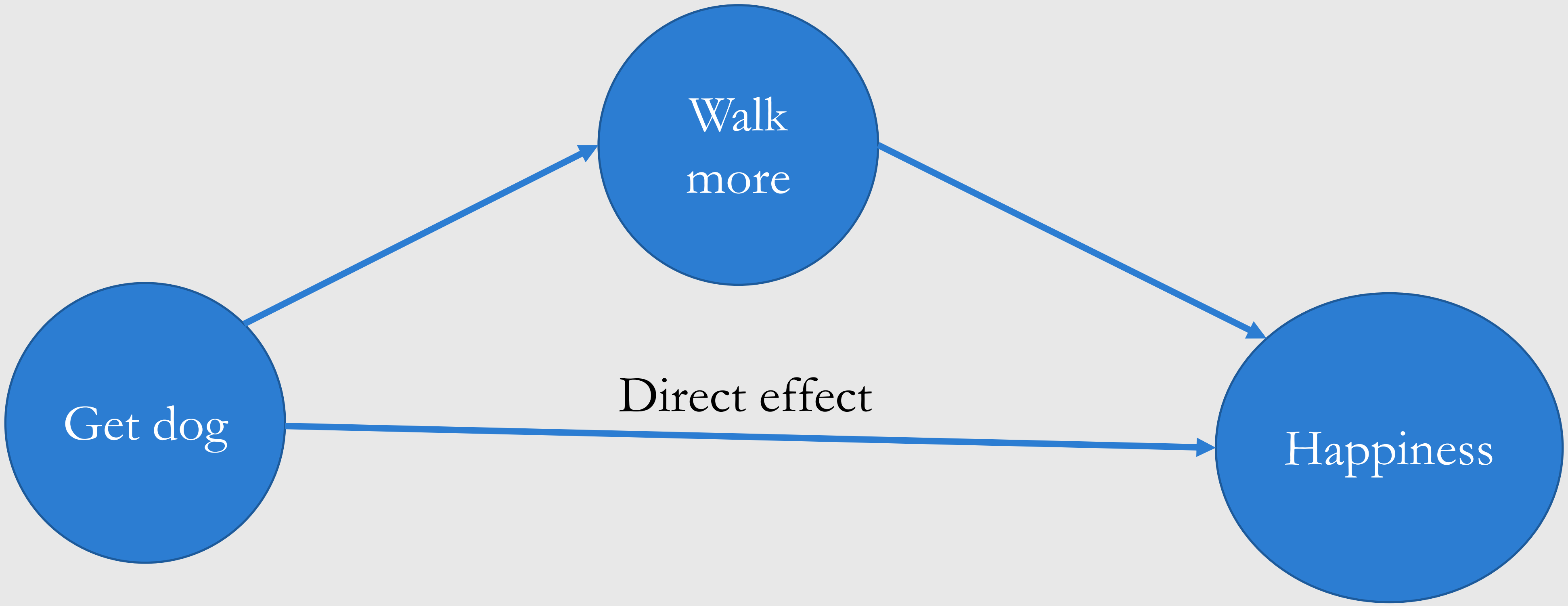
Important Application: Mediation

Mediation

Mediation



Mediation



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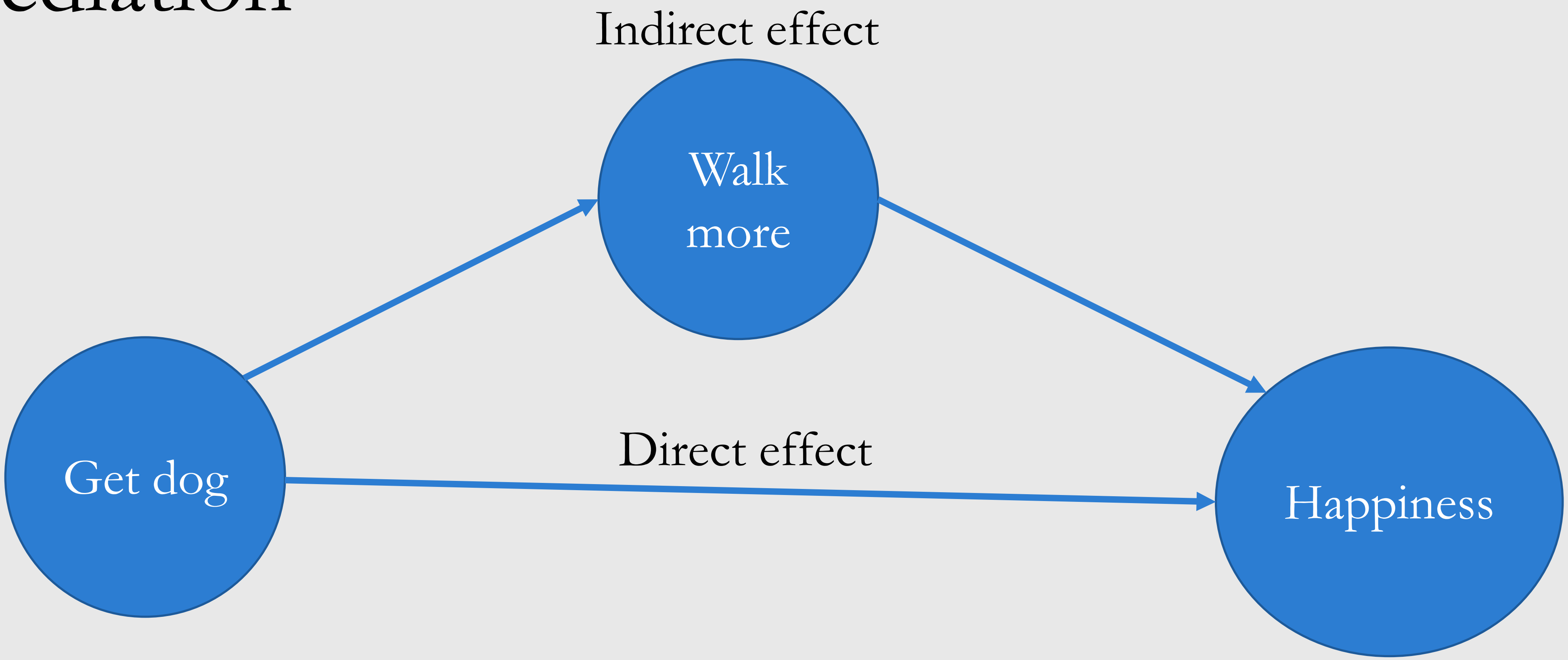

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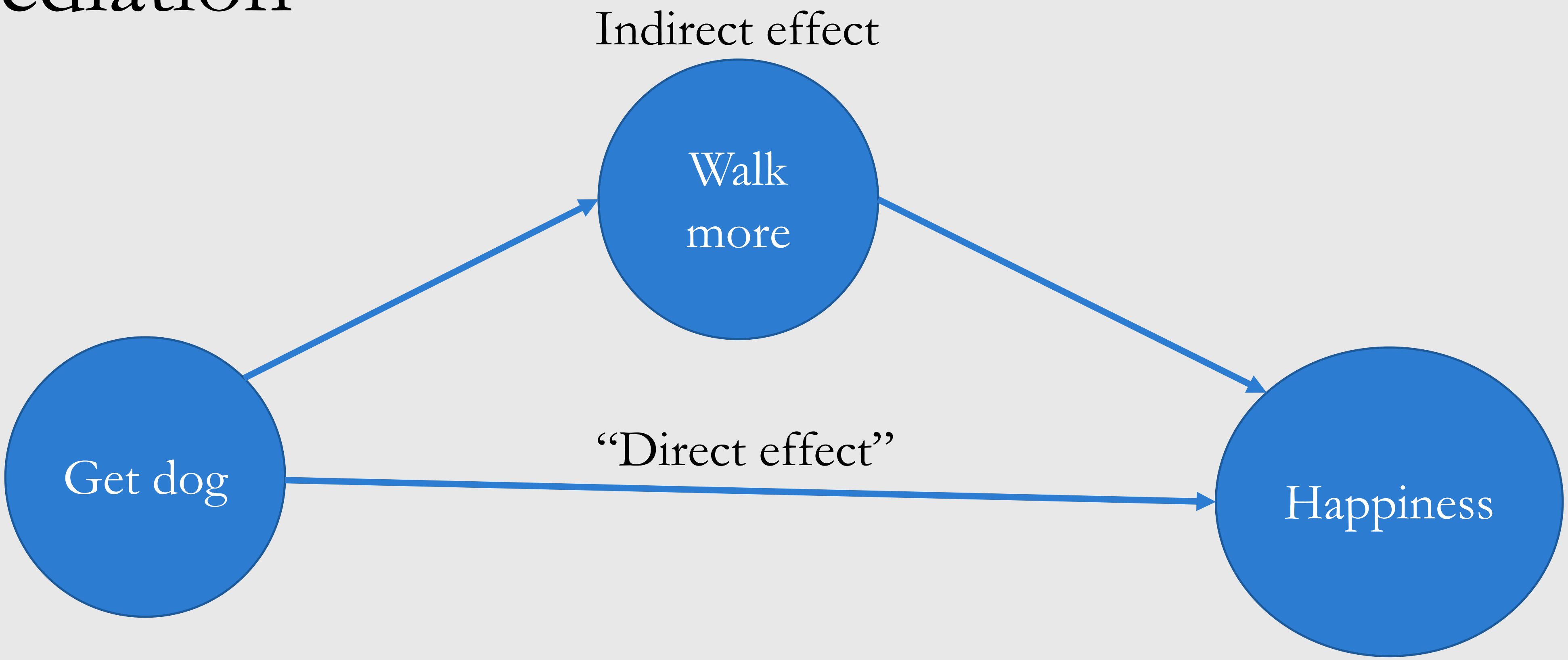

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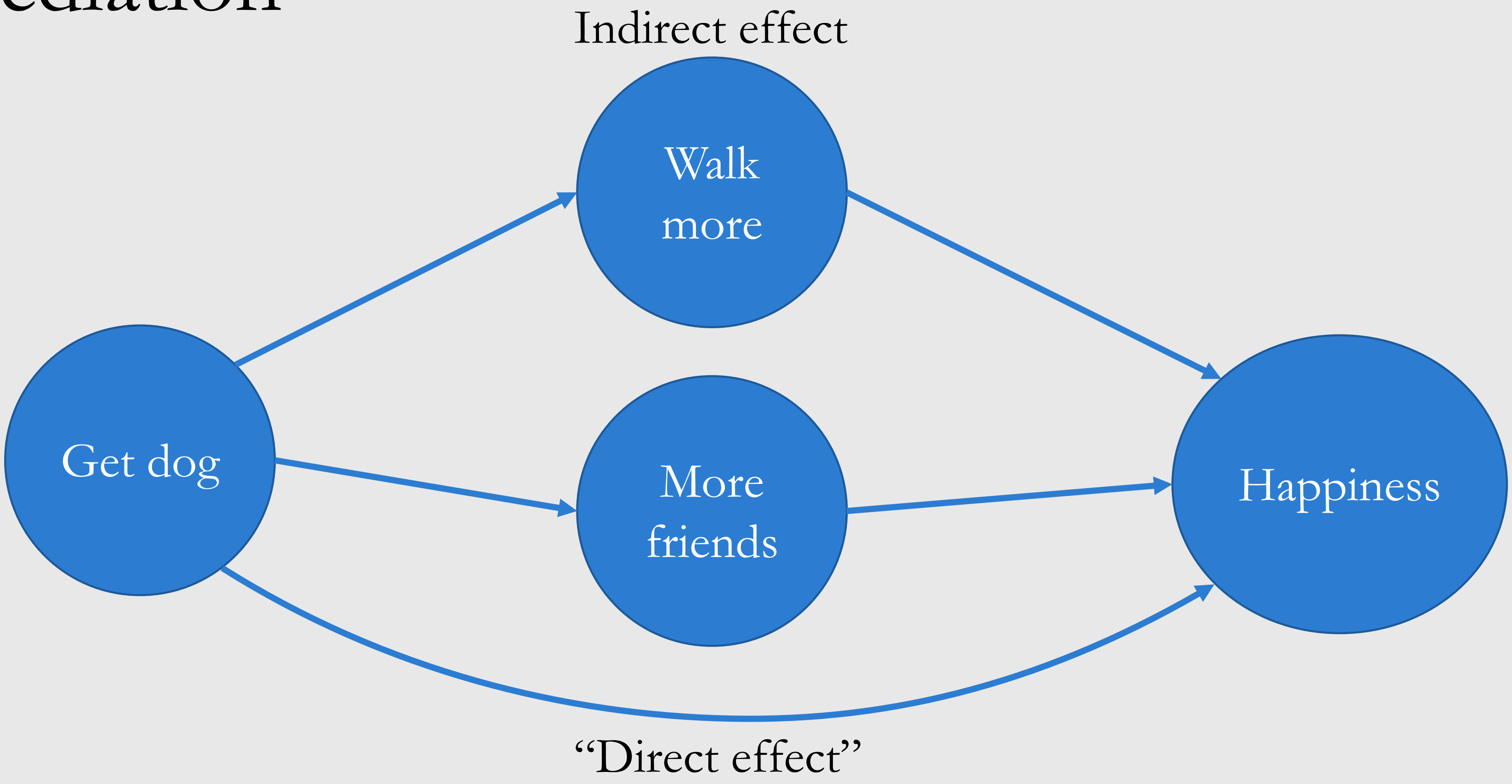
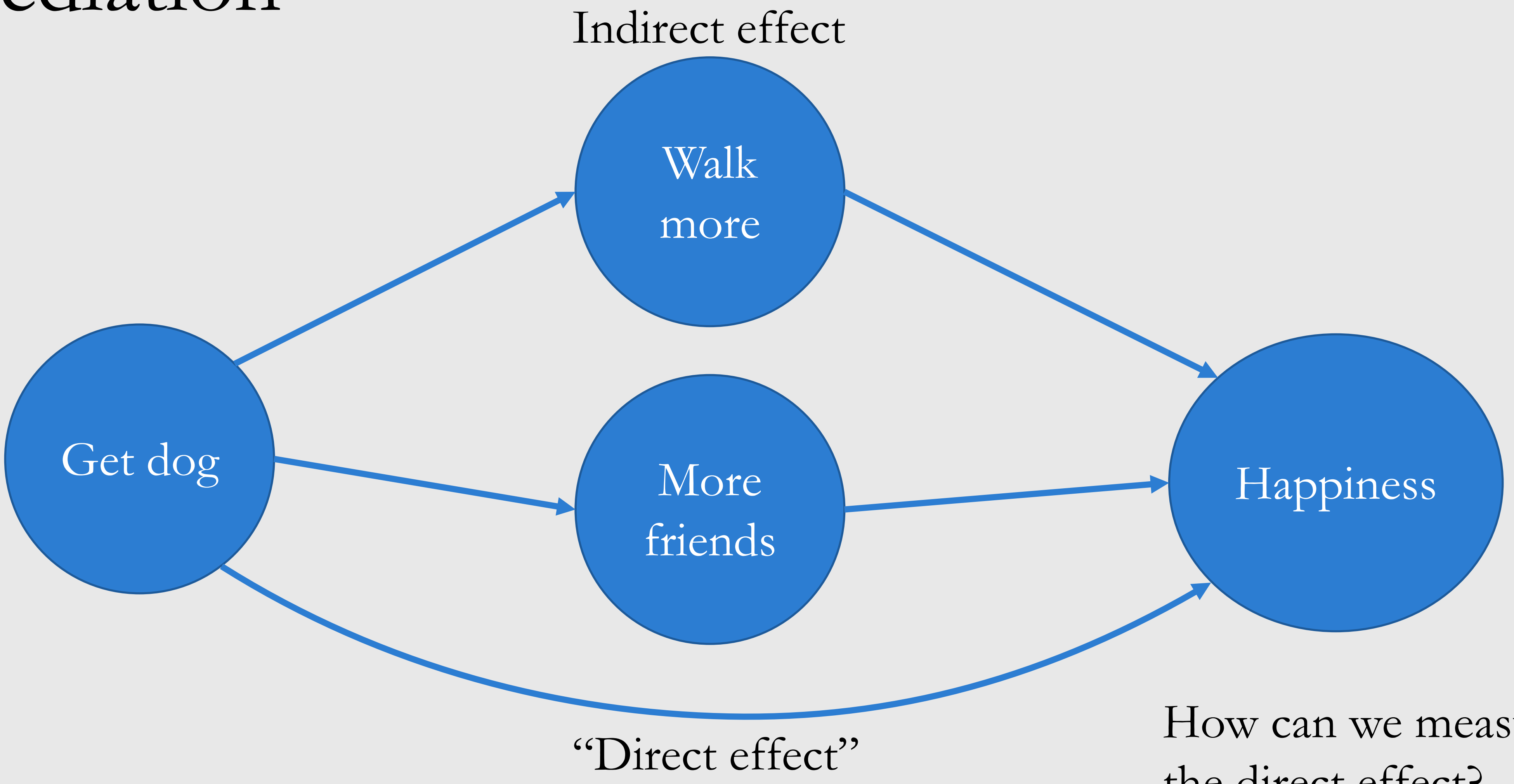
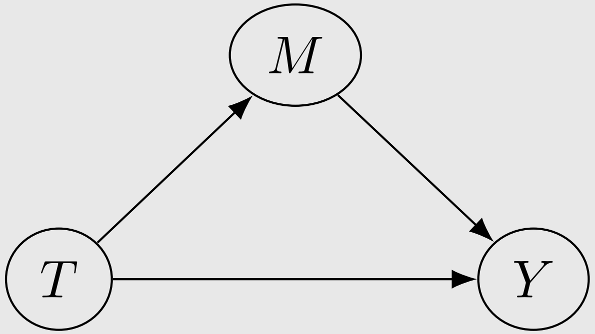

Indirect effect
Walk more
More friends
Get dog
Happiness
"Direct effect"

Mediation



How can we measure the direct effect?

Controlled Direct Effect (CDE)



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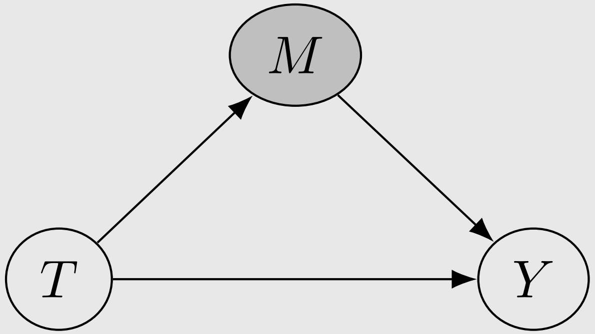
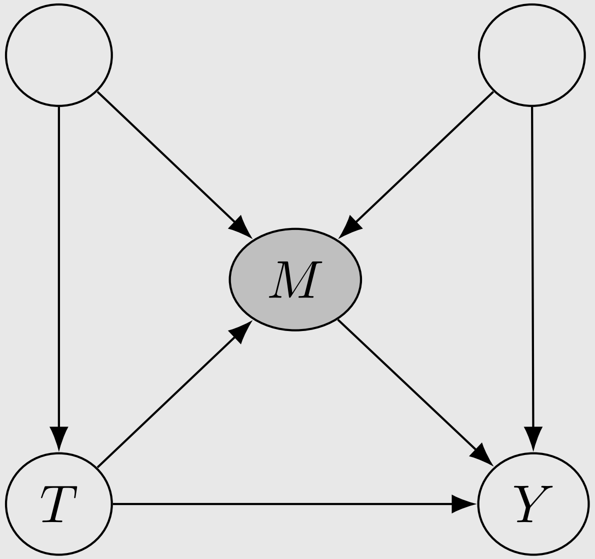


$$\mathbb{E}[Y \mid do(T=1), M=m] - \mathbb{E}[Y \mid do(T=0), M=m]$$

Controlled Direct Effect (CDE)



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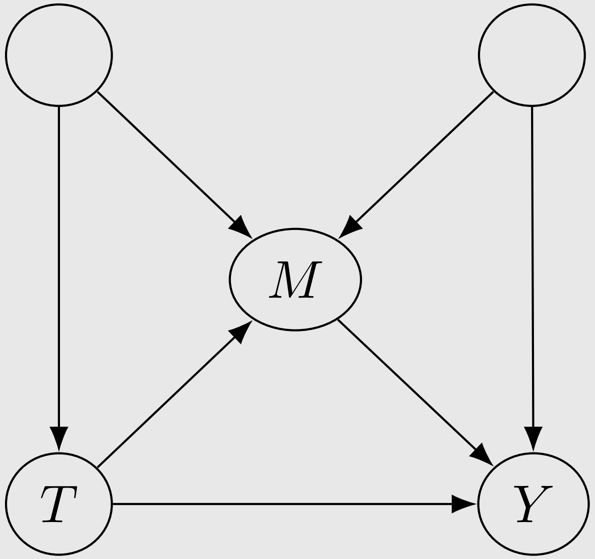

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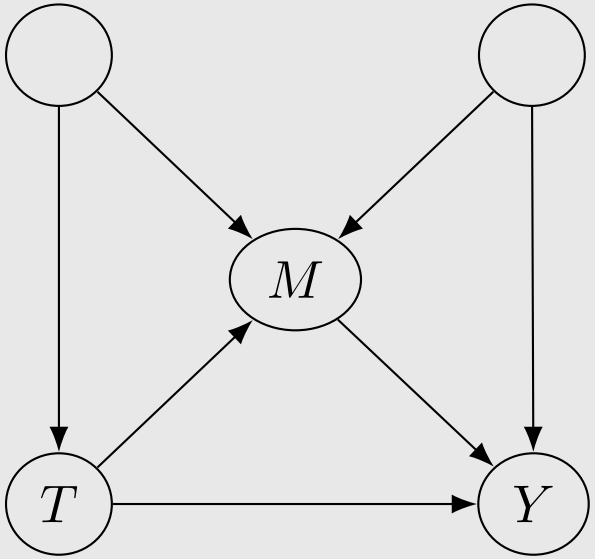

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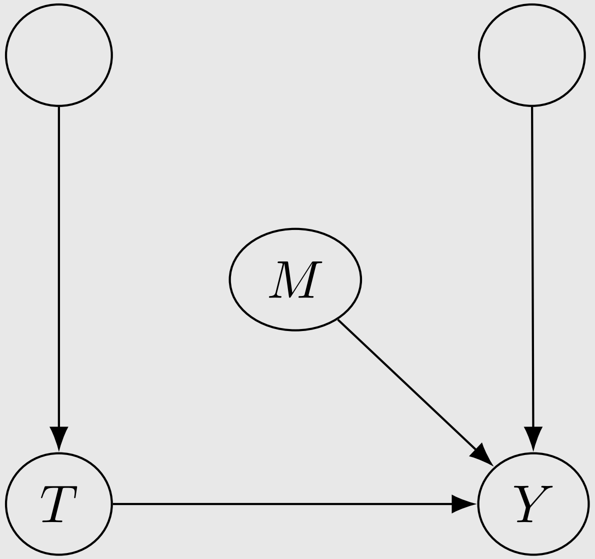

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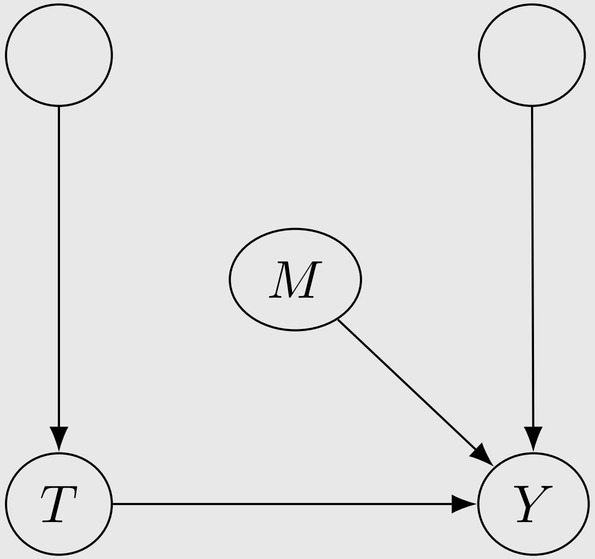

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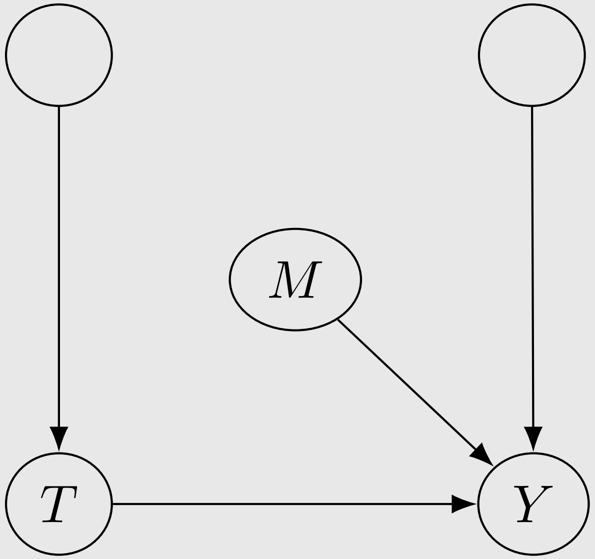

Problems:

- CDE is specific to the arbitrary choice of m

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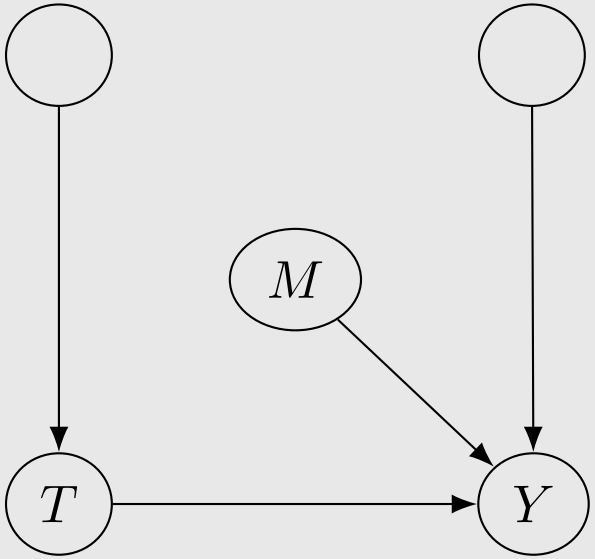

Problems:

- CDE is specific to the arbitrary choice of m
- How do we get the indirect effect? Can't just subtract the CDE from the total effect

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Natural Direct and Indirect Effects

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Subscript notation:

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Recall problems with CDE:

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Recall problems with CDE:

- CDE is specific to the arbitrary choice of m
- How do we get the indirect effect? Can't just subtract the CDE from the total effect

For example in linear setting, $\text{TE} = \text{NDE} + \text{NIE}$

Question:

Show that TE = NDE – NIE_r,
where $\text{NIE}_r \triangleq \mathbb{E}[Y_{1,M_0} - Y_{1,M_1}]$.

Comparison of Controlled vs. Natural Mediation

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CDE can always be measured via experiments (do-operator), but it has no clear undirect effect since there is no decomposition

NDE cannot always be measured via experiments since it is counterfactual, but it allows for the complete decomposition of the total effect into the NDE and NIE, which is what we'd like in mediation analysis

When We Can Measure NDE and NIE

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Adjustment set W

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When We Can Measure NDE and NIE

Adjustment set $\mathbb{W}$

Sufficient conditions for identifying NDE:

1. No member of $\mathbb{W}$ is a descendant of T
2. $\mathbb{W}$ blocks all backdoor paths from M to Y

$$\begin{aligned}\text{NDE} = \sum_m \sum_w & \left(\mathbb{E}[Y \mid do(T=1, M=m), W=w] - \mathbb{E}[Y \mid do(T=0, M=m), W=w]\right) \\ & \times P(M=m \mid do(T=0), W=w) P(W=w)\end{aligned}$$

When We Can Measure NDE and NIE

Adjustment set $\mathbb{W}$

Sufficient conditions for identifying NDE:

1. No member of $\mathbb{W}$ is a descendant of T
2. $\mathbb{W}$ blocks all backdoor paths from M to Y
3. $P(M = m \mid do(T = 0), W = w)$ is identifiable (e.g. no unblockable backdoor paths from T to M)
4.

$$\text{NDE} = \sum_m \sum_w \left(\mathbb{E}[Y \mid do(T = 1, M = m), W = w] - \mathbb{E}[Y \mid do(T = 0, M = m), W = w] \right)$$
$$\times P(M = m \mid do(T = 0), W = w) P(W = w)$$

When We Can Measure NDE and NIE

Adjustment set $\mathbb{W}$

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3. $P(M = m \mid do(T = 0), W = w)$ is identifiable (e.g. no unblockable backdoor paths from T to M)
4. $\mathbb{E}[Y \mid do(T = t, M = m), W = w]$ is identifiable (e.g. no unblockable backdoors paths from T to Y)

$$\begin{aligned}\text{NDE} = \sum_m \sum_w & (\mathbb{E}[Y \mid do(T = 1, M = m), W = w] - \mathbb{E}[Y \mid do(T = 0, M = m), W = w]) \\ & \times P(M = m \mid do(T = 0), W = w) P(W = w)\end{aligned}$$

When We Can Measure NDE and NIE

Adjustment set W

Sufficient conditions for identifying NDE:

1. No member of W is a descendant of T
2. W blocks all backdoor paths from M to Y
3. $P(M = m \mid do(T = 0), W = w)$ is identifiable (e.g. no unblockable backdoor paths from T to M)
4. $\mathbb{E}[Y \mid do(T = t, M = m), W = w]$ is identifiable (e.g. no unblockable backdoors paths from T to Y)

$$\begin{aligned} \text{NDE} &= \sum_m \sum_w (\mathbb{E}[Y \mid do(T = 1, M = m), W = w] - \mathbb{E}[Y \mid do(T = 0, M = m), W = w]) \\ &\quad \times P(M = m \mid do(T = 0), W = w) P(W = w) \\ &= \sum_m \sum_w (\mathbb{E}[Y \mid T = 1, M = m, W = w] - \mathbb{E}[Y \mid T = 0, M = m, W = w]) \\ &\quad \times P(M = m \mid T = 0, W = w) P(W = w) \end{aligned}$$

When We Can Measure NDE and NIE

Adjustment set W

Sufficient conditions for identifying NDE:

$$\text{TE} = \text{NDE} - \text{NIE}_r$$

1. No member of W is a descendant of T
2. W blocks all backdoor paths from M to Y
3. $P(M = m \mid do(T = 0), W = w)$ is identifiable (e.g. no unblockable backdoor paths from T to M)
4. $\mathbb{E}[Y \mid do(T = t, M = m), W = w]$ is identifiable (e.g. no unblockable backdoors paths from T to Y)

$$\begin{aligned}
\text{NDE} = &\sum_m \sum_w \left(\mathbb{E}[Y \mid do(T = 1, M = m), W = w] - \mathbb{E}[Y \mid do(T = 0, M = m), W = w]\right) \\
&\quad \times P(M = m \mid do(T = 0), W = w) P(W = w) \\
= &\sum_m \sum_w \left(\mathbb{E}[Y \mid T = 1, M = m, W = w] - \mathbb{E}[Y \mid T = 0, M = m, W = w]\right) \\
&\quad \times P(M = m \mid T = 0, W = w) P(W = w)
\end{aligned}$$

Question:

Come up with your own example of mediation and the corresponding graph. Then, determine whether you can identify the NDE and NIE from observational data.

Path-Specific Effects

Measure causal effects along arbitrary path or set of paths in the causal graph

See [“Identifiability of Path-Specific Effects” (Avin et al., 2005)]